Mathematical modeling and linear systems – Teacher Pages

Our Goal: To interactively understand a real-world situation and then connect this to the practical use of solving linear systems in mathematics.

Assumptions: Student (a) can graph a line from a linear equation in 2 variables in the standard form, (b) has seen and solved a 2x2 linear system in 2 variables using either the substitution or addition method and (c) knows how to solve use the distance, rate and time formula $s = vt$.

Problem situation: A boat travels downstream and upstream on a river with a current. We assume that – if the boat was in still water – its speed would be constant. We also assume that the speed of the current of the river is constant. **How does the speed of the current affect that actual speed of the boat as it travels downstream and as it travels upstream?**

Open the interactivity: [BoatCurrent](http://www.mathcasts.org/mtwiki/InterA/BoatCurrent) (online) **or boatcurrent.html** (offline).

Getting started: We are going to test the boat using the simulator. But first let's consider.
Let v_b = the speed of the boat in still water. Let v_r = speed of the river current (assume positive ☺).
Let v_D = speed of the boat downstream. Let v_U = speed of the boat upstream.
Fill in the squares at right with v_b, v_D and v_U.

- **Answer:** It is important that the students understand that the "throttle" is at the same position for **all three boats**. So going *with the current* "adds" the speed of the current and going *against the current* "subtracts" the speed of the current.
  You might want to mention biking or walking "with" and "against" the wind.

$$\boxed{v_U} < \boxed{v_b} < \boxed{v_D}$$

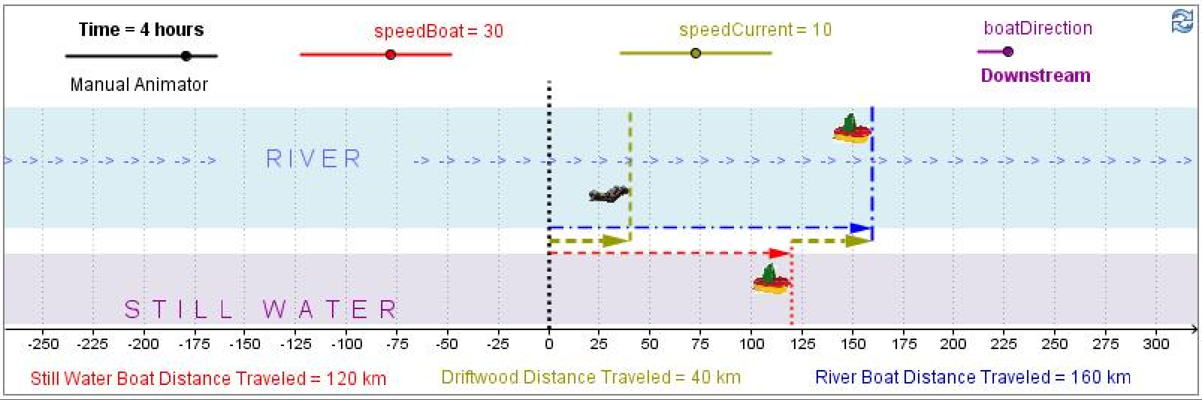
Downstream

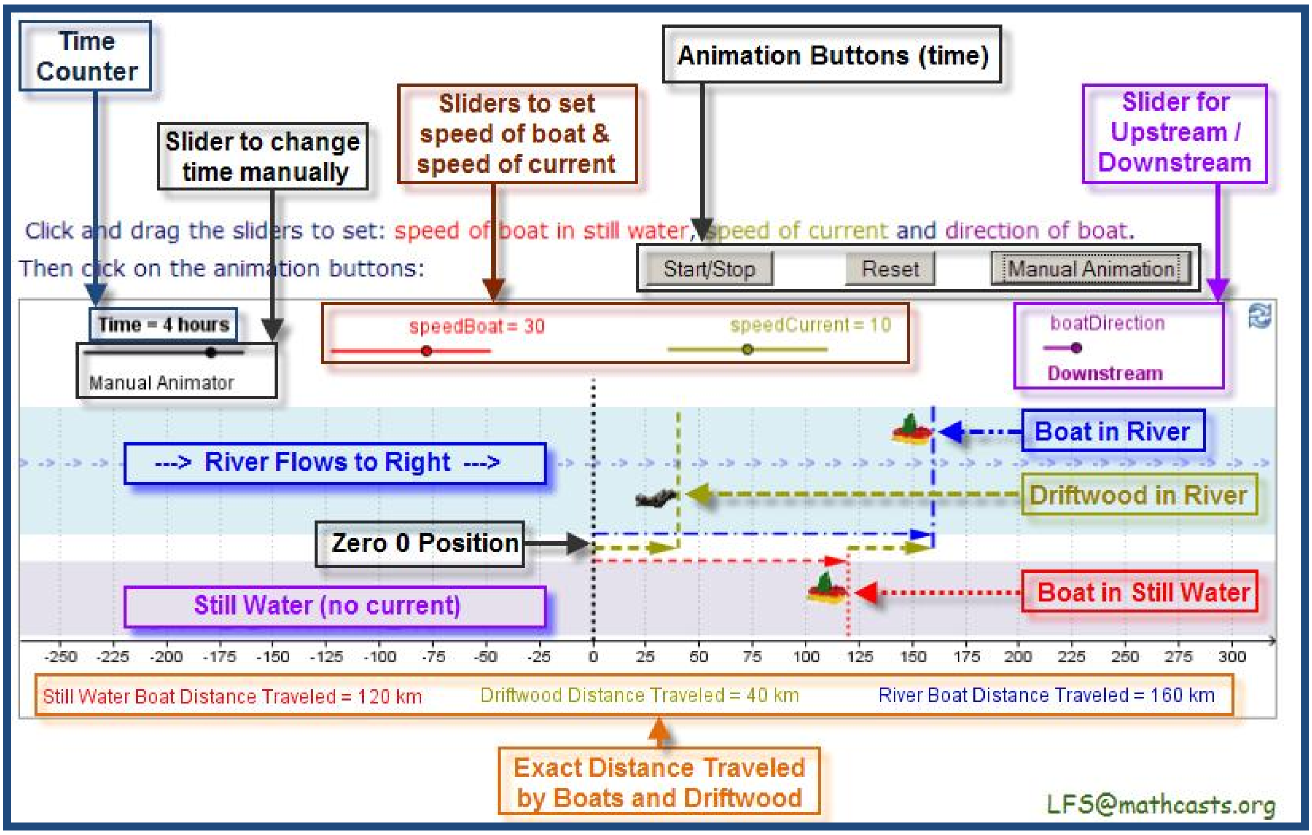
➢ Click on the Start/Stop button. As **time** advances (top left), the **boat** and the **driftwood** will move in the river with the current and the boat in still water will move.

Discussion points:

➢ Click again on the start/stop button to stop the animation. Look around at what has changed and what has stayed the same on the simulator.

- Here is what the simulator looks like at Time = 4 hours and below the parts are explained.





➢ Click again on the start/stop button and now let the animation finish. Notice **Time =5 hours**.

- A. Answers below assume the initial slider values of 30km/h and 10 km/h.
- B. Note that exact distances are given below the simulation.

➢ Find the distance traveled by the boat in still water in these 5 hours. Calculate its speed v_b. Does your answer match the slider speed at the top (just right of time)?

- **Answer:** This boat will have traveled **150km**. It speed is: $v_b = \frac{s}{t} = \frac{150\text{km}}{5\text{h}} = 30\,\text{km/h}$.

➢ Find the distance traveled by the driftwood. Calculate its speed v_r. Does it match the yellow slider? What can you say about the speed of the driftwood and the speed of the current?

- **Answer:** This boat will have traveled **50km**. It speed is: $v_r = \frac{s}{t} = \frac{50\text{km}}{5\text{h}} = 10\,\text{km/h}$.
- The **speed of the driftwood equals** the **speed of the current.**

➢ Finally, find the distance traveled by the boat going downstream. What can you say about this distance, the distance traveled by the boat in still water and the distance traveled by the driftwood.

- **Answer:** This boat will have traveled **200km**. We see that **200km=150km+50km** or with words:
- distance of boat going downstream **=** distance of boat in still water **+** distance of driftwood.

➢ Calculate the speed v_D of the boat going downstream. What is the **relationship** between this speed, the speed of the boat in still water and the speed the river current. Write this relationship as an equation using v_b, v_r and v_D and **circle your equation** so you can find it. $v_b + v_r = v_D$

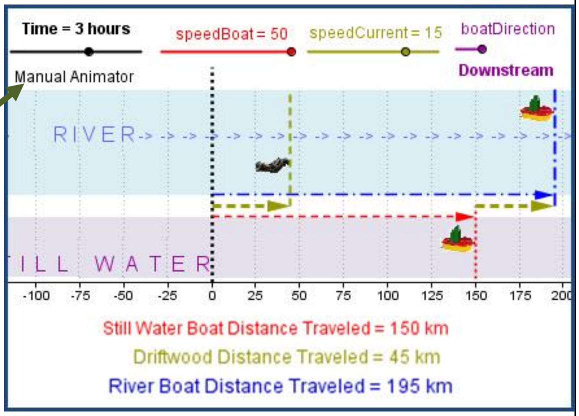
- It speed is: $v_D = \frac{s}{t} = \frac{200\text{km}}{5\text{h}} = 40\,\text{km/h}$.

- Click and drag sliders to change the speed of the boat in still water v_b = **50 km/h** and of the river current v_r = **15 km/h**. Using the formula: $s_D = (v_b + v_r)\cdot t$ calculate the distance the boat will travel downstream in **3 hours**.

- **Answer:** This boat will have traveled: $s_D = (50\frac{km}{h} + 15\frac{km}{h})\cdot 3h = (65\frac{km}{h})\cdot 3h = 195\,\text{km}$

- Click on Manual Animation button. You will see a slider underneath the time. Click and drag this slider to time **t=3 hours**. Find the distance traveled by the boat going downstream. Does this agree with your calculation?

Manual Animator

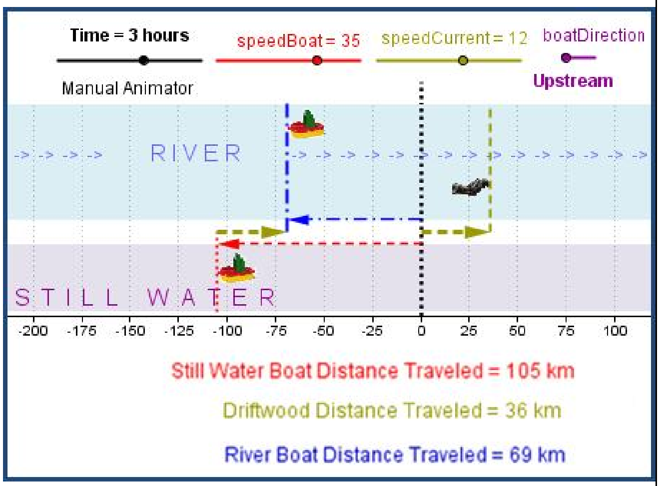


- Calculate this boats speed v_D. Does your equation using v_b, v_r and v_D work here? It should.

- **Answer:** $v_D = \frac{s}{t} = \frac{195\text{km}}{3\text{h}} = 65\frac{\text{km}}{h}$. Yes, since $v_b + v_r = 50\frac{km}{h} + 15\frac{km}{h} = 65\frac{km}{h} = v_D$.

Upstream

- If you want, change the slider values for speed of the boat in still water v_b and of the current v_r.
- Click and drag the slider at the right so that the boat in the river is going upstream. (Remember that even though the boat in still water will go in the opposite direction as before, this doesn't affect its speed since direction doesn't matter in still water.)
- Click on the reset button or click and drag the manual animation slider to t = 0.
- Run the animation – click on the Start/Stop button or use the manual animator.
- Find the distance traveled by the boat going upstream.



- We see that **69km=105km-36km** or with words:
- distance of boat going **upstream** = distance of boat in still water – distance of driftwood.

- Calculate its speed v_U. What is the **relationship** between this speed, the speed of the boat in still water and the speed of the current.
- **Answer:** Using v_b **=35 km/h**, v_r **=12 km/h** and **Time =3h**, we see: $s_U = 69\,\text{km}$.
- This means $v_U = \frac{s}{t} = \frac{69\text{km}}{3\text{h}} = 23\,\text{km/h}$. So $v_U = 23\frac{km}{h} = 35\frac{km}{h} - 12\frac{km}{h} = v_b - v_r$

- Write an equation using v_b, v_r and v_U and **circle your equation**: $v_b - v_r = v_U$

Modeling – translating real-world to mathematics, solving and translating back

Problem: A boat travels downstream on a river at 60 km/h. It travels upstream on the same river at 40 km/h. What is the speed of the boat in still water and the speed of the current of the river? Assume that the speed of the boat in still water and speed of the current of the river are constant.

- What are the values of v_b, v_r, v_D and v_U? If they are unknown variables, write " = ? "

- **Answer:** v_D =60 km/h v_U =40 km/h v_b = ? v_r = ? (these are the unknown variables)

- Using your circled equations from above, make a system of 2 equations in 2 unknown variables. Solve this system and write down your answer.

- **Answer:** $\begin{cases} v_b + v_r = v_D \\ v_b - v_r = v_U \end{cases}$ With our numbers we have: $\begin{cases} v_b + v_r = 60 \\ v_b - v_r = 40 \end{cases}$. This is our system!

We solve below using the addition method.

$$\begin{cases} v_b + v_r = 60 \\ v_b - v_r = 40 \end{cases} \Rightarrow \begin{cases} v_b + v_r = 60 \\ 2v_b = 100 \end{cases} \Rightarrow \begin{cases} v_b + v_r = 60 \\ v_b = 50 \end{cases} \Rightarrow \begin{cases} 50 + v_r = 60 \\ v_b = 50 \end{cases} \Rightarrow \begin{cases} v_r = 60 - 50 \\ v_b = 50 \end{cases} \Rightarrow \begin{cases} v_r = 10 \\ v_b = 50 \end{cases}$$

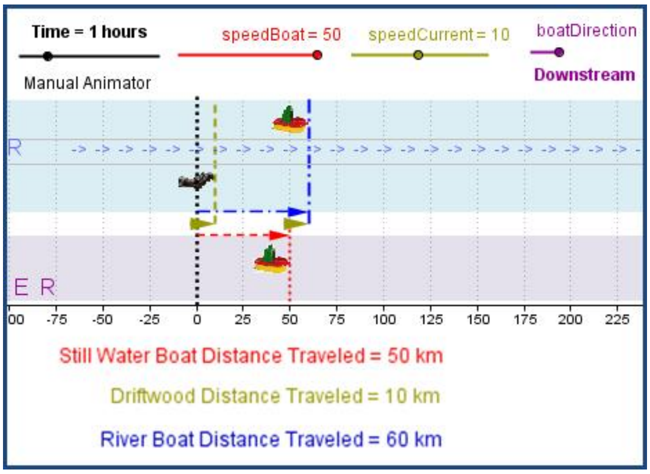
- Rewrite this answer as an answer that fits the question – write in people language.

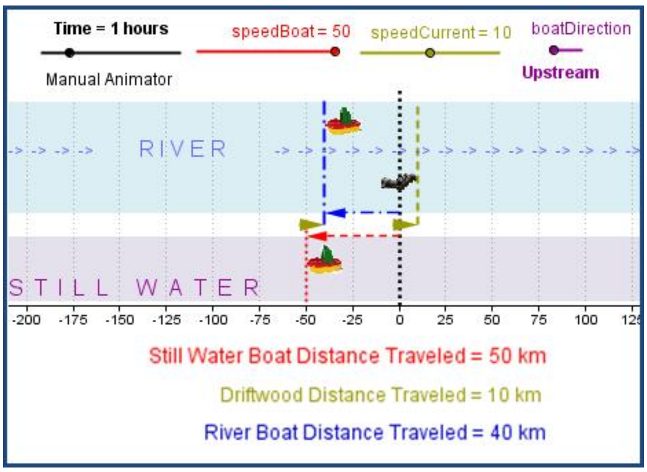
- **Answer:** The speed of the boat in still water is: v_b =50 km/h and
  the speed of the current is: v_r =10 km/h.

- Think a bit about how you would check your answer on the simulator. What should be the values on the sliders for v_b and v_r? Change the sliders to these values.

- Run the simulator **downstream** for t=1 hour (manually). If your answer is correct, how far should the boat going downstream travel? If you can, take a screenshot and explain how this shows that your answer is correct.

- **Answer: Below left:** River Boat Distance Traveled **downstream** is 60 km as given in problem.





- Run the simulator **upstream** for t=1 hour (manually). If your answer is correct, how far should the boat going upstream travel? If you can, take a screenshot and explain it.

- **Answer - Above right:** River Boat Distance Traveled **upstream** is 40 km as given in problem.

A little challenge:

Problem: A boat went 20km downstream in 20 minutes. It then went upstream 40km in 2 hours. At what speed would a piece of driftwood move in this river? Assume that the speed of the boat in still water and speed of the current of the river are constant.

Solve this problem and then check in on the simulator. Take screenshots of your 2 checks (downstream and upstream) and explain them.

- **Answer:** Let's model (translate the problem into mathematics).

1. A boat went 20km downstream in 20 minutes means:

$$v_D = \frac{20\,\text{km}}{20\,\text{min}} = \frac{20\,\text{km}}{20\,\text{min}} \cdot \frac{60\,\text{min}}{1\text{h}} = \frac{20\,\text{km}}{1} \cdot \frac{3}{1\text{h}} = 60\,km/h$$

2. upstream 40km in 2 hours means:

$$v_U = \frac{40\,\text{km}}{2h} = 20\,km/h$$

The same equations apply as above

$\begin{cases} v_b + v_r = v_D \\ v_b - v_r = v_U \end{cases}$ With our numbers we have: $\begin{cases} v_b + v_r = 60 \\ v_b - v_r = 20 \end{cases}$. This is our system!

$$\begin{cases} v_b + v_r = 60 \\ v_b - v_r = 20 \end{cases} \Rightarrow \begin{cases} v_b + v_r = 60 \\ 2v_b = 80 \end{cases} \Rightarrow \begin{cases} v_b + v_r = 60 \\ v_b = 40 \end{cases} \Rightarrow \begin{cases} 40 + v_r = 60 \\ v_b = 40 \end{cases} \Rightarrow \begin{cases} v_r = 60 - 40 \\ v_b = 40 \end{cases} \Rightarrow \begin{cases} v_r = 20 \\ v_b = 40 \end{cases}$$

- **Answer:** The speed of the boat in still water is: v_b =40 km/h and
  the speed of the current is: v_r =20 km/h.

Driftwood moves at the speed of the current so: A piece of driftwood would move at v_r =20 km/h.

Note: In order for the sliders to give even values, it is impossible for 20 min =1/3 hour to be displayed on the time slider so students will need to think in terms of one hour.